Optimization Theory and Methods

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同济经管
TONGJI SEM

魏可佶 kejiwei@tongji.edu.cn
https://kejiwei.github.io/

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- Standard form of a Linear Programming Problem (LP)
- Useful terms
- Solution algorithms
- Optimality conditions
- Production example
- Sensitivity analysis

$$\boldsymbol{MIN}\ (c_1x_1 + c_2x_2 + \cdots + c_nx_n)$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

… … … …

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$x_i \geq 0, \forall i \in \{1,2,\dots,n\}$$

↳ Production Example

(1) A company produces 3 products. Each unit of product 1, 2, and 3 generates a profit of $10, $12 and $12 respectively.

(2) Each product has to go through a manufacturing, assembly, and testing phase.

(3) The company's resources are such that only 20 hours of manufacturing, 20 hours of assembly, and 20 hours of testing are available.

(4) Each unit of product 1 has to spend 1 hr in manufacturing, 2 hrs in assembly, and 2 hrs in testing.

↳ Production Example (cont.)

(5) Each unit of product 2 has to spend 2 hrs in manufacturing, 1 hr in assembly, and 2 hrs in testing.

(6) Each unit of product 3 has to spend 2 hrs in manufacturing, 2 hrs in assembly, and 1 hr in testing.

(7) Company wants to know how many units of each product it should produce, in order to maximize its profit.

↳ Production Example (cont.)

$MAX\ (10x_1 + 12x_2 + 12x_3)$

s.t.

$x_1 + 2x_2 + 2x_3 \leq 20$

$2x_1 + x_2 + 2x_3 \leq 20$

$2x_1 + 2x_2 + x_3 \leq 20$

$x_1, x_2, x_3 \geq 0$

In Standard Form

$MIN\ (-10x_1 - 12x_2 - 12x_3)$

s.t.

$x_1 + 2x_2 + 2x_3 + x_4 = 20$

$2x_1 + x_2 + 2x_3 + x_5 = 20$

$2x_1 + 2x_2 + x_3 + x_6 = 20$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

↳ Basic Solutions

- If a standard form LP has an optimal solution, it must also have an optimal basic solution
  - A basic solution is one in which at most m variables take on non-zero values
  - These m variables are referred to as basic variables (note that basic variables can also take on value 0)
    - $n - m$ non-basic variables
- A basic solution that satisfies all constraints is called a basic feasible solution.

A basic feasible solution is a corner of the feasible region.

↳ Dual Variables

- The dual variable value of a constraint can be interpreted as the value of relaxing the constraint per unit relaxation (without changing the other constraints)
- Also called shadow prices
- There is one dual variable associated with each constraint
  - Indicates how much a constraint matters
  - If the constraint is not binding, the dual value is equal to 0 and relaxing it by 1 unit has no effect on optimal solution.
- Let us denote the dual variable values by a vector π
  - Dimension of π is $m * 1$: one dual variable per constraint

- Reduced cost of variable x_i is:

$$c_i - a_{1i}\pi_1 - a_{2i}\pi_2 - \cdots - a_{mi}\pi_m$$

- Reduced cost of a variable x_i can be viewed as an estimate of the rate of change in the objective function value per unit increase in x_i while satisfying all constraints.

- Reduced cost = 0 of any basic variable x_i

↳ Example: Reduced Cost Calculation

	x_1	x_2	x_3	x_4	x_5	x_6	b	Dual
$Cost(j)$	-10	-12	-12	0	0	0		
$A(i,j)$	1	2	2	1	0	0	20	-3.6
	2	1	2	0	1	0	20	-1.6
	2	2	1	0	0	1	20	-1.6

- Reduced cost $x(j) = \mathrm{c(j)} - \mathrm{a(1,j)} * \pi_1 - a(2,j) * \pi_2 - a(3,j) * \pi_3$
- Reduced cost $x(1) = -10 - 1 * (-3.6) - 2 * (-1.6) - 2 * (-1.6) = 0$
- Reduced cost $x(2) = -12 - 2 * (-3.6) - 1 * (-1.6) - 2 * (-1.6) = 0$
- Reduced cost $x(3) = -12 - 2 * (-3.6) - 2 * (-1.6) - 1 * (-1.6) = 0$
- Reduced cost $x(4) = 0 - 1 * (-3.6) - 0 - 0 = 3.6$
- Reduced cost $x(5) = 0 - 0 - 1 * (-1.6) - 0 = 1.6$
- Reduced cost $x(6) = 0 - 0 - 0 - 1 * (-1.6) = 1.6$

- Many algorithms can be used to solve the LP
- Simplex algorithm
  - Searches for an optimal solution by moving from one basic feasible solution to another, along the edges of the feasible region, in direction of cost decrease (graphically, moves from corner to corner)
- Interior point methods
  - Approach the optimal solution through the interior of the convex polygon. Examples include:
    - Affine scaling
    - Log barrier methods
    - Primal-dual methods

- Given an initial basic solution:

1 – Compute the reduced costs of all non-basic variables. If they are all non-negative, stop.

2 – If not, choose some non-basic variable with negative reduced cost. (It will now become a basic variable)

3 – Identify a basic variable and set it to 0. (It will now become a non-basic variable)

4 – Solve for the value of the new set of basic variables.

5 – Solve for the new values of the dual variables.

6 – Return to Step 1.

↳ Simplex Optimality Conditions

- Basic feasible solution $\boldsymbol{x}$ (and dual solution $\boldsymbol{\pi}$) is optimal when:
  - The reduced costs of all basic variables equal 0
    - Maintained at each iteration of the simplex algorithm
  - The reduced costs of all non-basic variables are non-negative
    - Not necessarily maintained at each iteration of the simplex algorithm
  - Complementary slackness is satisfied (maintained at each iteration of the simplex algorithm)
    - Dual variable value is zero unless its associated constraint is binding: automatically holds for a standard form problem.
    - The product of the value of the decision variable x_i and the value of its reduced cost is always zero.
      - So variable value is zero OR reduced cost is zero OR both.
      - Maintained at each iteration of the simplex algorithm.

↳ Example: A Basic Feasible Solution

For the standard form problem representation:

$$\begin{aligned} &MIN\ (-10x_1 - 12x_2 - 12x_3) \\ &\text{s.t.} \\ &x_1 + 2x_2 + 2x_3 + x_4 = 20 \\ &2x_1 + x_2 + 2x_3 + x_5 = 20 \\ &2x_1 + 2x_2 + x_3 + x_6 = 20 \\ &x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

- **Basic vs. non-basic variables**
  - 3 constraints => 3 basic variables
  - x_4, x_5, x_6 are non-basic variables => $x_4, x_5, x_6 = 0$
  - $x_1, x_2, x_3 > 0$ => x_1, x_2, x_3 are basic
- **Dual values**
  - All constraints are trivially binding because they are all equality constraints.
- **Reduced costs**
  - Reduced costs of x_1, x_2, x_3 (basic variables) equal 0.
  - Reduced costs of x_4, x_5, x_6 (non-basic variables) are ≥ 0.
  - Solution is optimal because all reduced costs (basic + non-basic) are ≥ 0.

Variable	Value	Reduced Cost
x_1	4	0
x_2	4	0
x_3	4	0
x_4	0	3.6
x_5	0	1.6
x_6	0	1.6

- Given an initial basic solution:

1 – Compute the reduced costs of all non-basic variables. If they are all non-negative, stop.

2 – If not, choose some non-basic variable with negative reduced cost. (It will now become a basic variable)

3 – Identify a basic variable and set it to 0. (It will now become a non-basic variable)

4 – Solve for the value of the new set of basic variables.

5 – Solve for the new values of the dual variables.

6 – Return to Step 1.

- How does the objective function value and optimality conditions change when:
  - A new constraint is added
  - A new variable is added
  - The cost coefficient of a non-basic variable changes
  - The constraint coefficient of a non-basic variable changes

- If the current solution satisfies the new constraint, the current solution is optimal. (Why?)
- Otherwise re-solve.
- Example:
  - Each product of types 1 and 2 require 2 hours of time from specially qualified testers, and company resources allow for only 15 hrs of these specialized testers.
  - New inequality constraint: $2x_1 + 2x_2 \leq 15$.
  - At current solution, this is violated because $2 * 4 + 2 * 4 = 16 > 15$.

$\Rightarrow$ Re-solve after adding the new constraint.

- Feasibility of the current solution is not affected. (Why?)
- But we must check whether the same solution is still optimal.
- Condition: all reduced costs ≥ 0
- Only need to check the reduced cost of the new variable (Why?)

$$= c_{new} - \sum_{i=1}^{m} a_{i,new}\pi_i$$

- If it is ≥ 0, the current solution remains optimal.
- Else, the current solution is no longer optimal. The new variable becomes a basic variable at the next Simplex iteration.

↳ Back to the Production Example

- Company is thinking of introducing a new product. The new product would generate a profit of $11/unit. It would require 2 hrs of manufacturing, 2 hrs of assembly, and 2 hrs of testing. Should they introduce it?
- Calculate the reduced cost of the new product:
  - Reduced cost (New)= $-11 - 2\,(-3.6) - 2(-1.6) - 2(-1.6) = 2.6 \geq 0$

  $\Rightarrow$DO NOT introduce the product.
- **What if the product generated $14 of profit instead?**
  - Reduced cost (New)= $-14 - 2(-3.6) - 2(-1.6) - 2(-1.6) = -0.4 < 0$

  $\Rightarrow$Solution can be improved by introducing the new product.

  $\Rightarrow$Re-solve the problem to get the new optimal solution.

↳ Cost Coefficient of a Non-Basic Variable Changes

- The cost coefficient of variable x_v changes from C_v to $C_v + \Delta$.
  - Δ can be positive or negative.
- Feasibility of current solution not affected. (Why?)
- Check optimality conditions, i.e. whether reduced costs are non-negative.
- The only reduced cost that gets modified is that of x_v. (Why?)
- Let $C_v{}'$ be the current reduced cost of x_v.
- New reduced cost is $C_v{}' + \Delta$. (Why?)
- If $C_v{}' + \Delta \geq 0$ => Current solution is still optimal.
- Else, current solution is no longer optimal => Re-solve.

↳ Cost Coefficient of a Non-Basic Variable Changes(e.g.)

- **Example:**

$$\boldsymbol{Min}\ (-5x_1 - x_2 + 12x_3)$$

s.t.

$$3x_1 + 2x_2 + x_3 = 10$$
$$5x_1 + 3x_2 + x_4 = 16$$
$$x_1, x_2, x_3, x_4 \geq 0$$

- At a basic optimal solution: $\pi_1 = 10$, $\pi_2 = -7$
- Reduced cost of nonbasic variable $x_3 = 12 - 1(10) - 0(-7) = 2$

(1) Cost of x_3 increases from 12 to 15

- New reduced cost $= 5$ **=>** Current solution still optimal.

(2) Cost of x_3 decreases from 12 to 11

- New reduced cost $= 1$ **=>** Current solution still optimal.

(3) Cost of x_3 decreases from 12 to 6

- New reduced cost $= -4$ **=>** Current solution no longer optimal.

↳ Cost Coefficient of a Non-Basic Variable Changes

- Feasibility not affected. (Why?)
- Check optimality conditions i.e. whether reduced cost is still non-negative.
- Only the reduced cost of that non-basic variable is affected. (Why?)
- If the coefficient of the v^{th} variable in the z^{th} constraint changes by α,
  - Let $C_v{}'$ be the current reduced cost of x_v.
  - New reduced cost $= C_v{}' - \alpha\pi_z$. (Why?)
  - If $\alpha\pi_z \leq C_v{}'$ => Current solution is still optimal.
  - If $\alpha\pi_z > C_v{}'$ => Current solution no longer optimal. => Re-solve.

Cost Coefficient of a Non-Basic Variable Changes(e.g. Contd…)

- **Example:**

$$Min\ (-5x_1 - x_2 + 12x_3)$$

s.t.

$$3x_1 + 2x_2 + x_3 = 10$$
$$5x_1 + 3x_2 + x_4 = 16$$
$$x_1, x_2, x_3, x_4 \geq 0$$

- Current reduced cost of x_3 is $C_v{}' = 2$; $\pi_1 = 10$, $\pi_2 = -7$

(1) Change coefficient in constraint 1 from 1 to -1 $(\alpha = -2)$

- $C_v{}' - \alpha\pi_z = 2 - (-2)(10) = 22 \geq 0$

$\Rightarrow$ Current solution is still optimal.

(2) Change coefficient in constraint 1 from 1 to 2 $(\alpha = 1)$

- $C_v{}' - \alpha\pi_z = 2 - (1)(10) = -8 < 0$

$\Rightarrow$ Current solution is no longer optimal.

$\Rightarrow$ Re-solve.

Objective :

Key Concepts :